

FUNCTIONAL EQUATIONS FOR QUANTUM THETA FUNCTIONS

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Abstract. Quantum theta functions were introduced by the author in [Ma1]. They are certain elements in the function rings of quantum tori. By definition, they satisfy a version of the classical functional equations involving shifts by the multiplicative periods. This paper shows that for a certain subclass of period lattices (compatible with the quantization form), quantum thetas satisfy an analog of another classical functional equation related to an action of the metaplectic group upon the (half of) the period matrix. In the quantum case, this is replaced by the action of the special orthogonal group on the quantization form, which provides Morita equivalent tori. The argument uses Rieffel's approach to the construction of (strong) Morita equivalence bimodules and the associativity of Rieffel's scalar products.

§0. Introduction and summary

0.1. Theta functions and theta vectors. This paper is a contribution to the theory of quantum theta functions introduced in [Ma1] and further studied in [Ma2], [Ma3]. It addresses two interrelated questions:

- (a) *What is the connection between quantum theta functions and theta vectors?* This question was repeatedly raised by A. S. Schwarz, see e. g. [Sch].
- (b) *Does there exist a quantum analog of the classical functional equation for thetas (related to the action of the metaplectic group, see e. g. [Mu], §8)?*

Briefly, the (partial) answers we give here look as follows.

(i) Schwarz's theta vectors are certain elements of projective modules over C^∞ - or C^* -rings of unitary quantum tori. When such a module is induced from the basic Heisenberg representation by a lattice embedding into a vector Heisenberg group, the respective theta vectors f_T are parametrized by the points T of Siegel upper half space, and in different models of the basic representation take the form of a "quadratic exponent" $e^{\pi i x^t T x}$, a classical theta, or Fock's vacuum state: see Theorem 2.2 in [Mu].

To the contrary, quantum thetas are certain elements of the C^∞ function ring itself. (For this reason, partial multiplication of quantum thetas studied in [Ma3] does not seem to be directly related to the tensor product of projective (bi)modules).

The basic relationship between the two classes of objects is this. For a lattice embedded in a vector Heisenberg group, Rieffel's scalar products of *theta vectors*

(these products take values in the C^∞ ring of the relevant quantum torus) are certain *quantum theta functions*. This extends a calculation of Section 3 in [Ma4], which in turn generalized a result of [Bo]: see Theorems 3.2.1 and 3.6.1 below. Theorem 3.5.1 characterizes in abstract terms the subclass of quantum thetas that can be obtained in this way.

(ii) The classical functional equation relates two thetas considered as sections of line bundles over two isomorphic complex tori (Fourier series). Bundles and sections are lifted to the universal covers which are then identified compatibly with period lattices.

Similarly, the functional equation for scalar product quantum thetas stated here relates two theta functions in two quantum tori algebras related by a bimodule inducing their Morita equivalence. The equation then simply says that the respective thetas coincide after being applied to appropriate vacuum vectors, and becomes a particular case of Rieffel's associativity relations: see the Theorem 3.3.1.

Here are some details. Consider a classical theta function of $z \in \mathbf{C}^N$

$$\theta(z, \Omega) := \sum_{n \in \mathbf{Z}^N} e^{\pi i n^t \Omega n + 2\pi i n^t z} \quad (0.1)$$

where Ω is a symmetric complex matrix with positive defined imaginary part. This function satisfies two sets of functional equations. Firstly, for all $m \in \mathbf{Z}^N$,

$$\theta(z + m, \Omega) = \theta(z, \Omega), \quad (0.2)$$

$$\theta(z + \Omega m, \Omega) = e^{-\pi i m^t \Omega m - 2\pi i m^t z} \theta(z, \Omega). \quad (0.3)$$

Secondly,

$$\theta(\Omega^{-1}z, -\Omega^{-1}) = (\det(\Omega/i))^{1/2} e^{\pi i z^t \Omega^{-1} z} \theta(z, \Omega). \quad (0.4)$$

In fact, (0.4) is the most important special case of a more general modular functional equation related to the action of $Sp(2, \mathbf{Z})$ upon the space of pairs (z, Ω) which we do not spell out here.

The geometric meaning of these equations can be described as follows. Consider $\theta(z, \Omega)$ as a global section of the trivial line bundle over \mathbf{C}^N . Equations (0.2) allow us to consider it as a global section of the trivial line bundle over $(\mathbf{C}^*)^N$ as well. This section is written as a Laurent series in the basic characters $e(n) := e^{2\pi i n^t z}$ of $(\mathbf{C}^*)^N$. Equations (0.3) allow us to descend one step further, now turning θ into a section of a nontrivial line bundle \mathcal{L} on the complex torus \mathbf{C}^N/D where D is the sublattice generated by the unit vectors and the columns of Ω . This is achieved by embedding D into a vector Heisenberg group acting upon $\mathbf{C}^N \times \mathbf{C}$ compatibly with the projection, and then taking the quotient of this space with respect to D : see e. g. [Mu], p. 35. Now, changing the initial basis of \mathbf{C}^N (e. g. replacing it by the columns of Ω) produces an isomorphic triple $(\mathcal{T}, \mathcal{L}, \theta)$ consisting of a complex

torus, line bundle, and its section. This is the source of equation (0.4) and its generalizations.

Now deform the multiplication rule of the characters $e(n)$ by choosing an antisymmetric real matrix A and putting $e_\alpha(m)e_\alpha(n) := \alpha(m, n)e_\alpha(m+n)$ where $\alpha(m, n) = e^{2\pi i m^t A n}$. The deformed characters generate various function rings representing the quantum torus $T(\mathbf{Z}^N, \alpha)$ which should be considered as a deformation of $(\mathbf{C}^*)^N$ or of its maximal compact subtorus. The series (0.1) in which $e^{2\pi i n^t z}$ is replaced by $e_\alpha(n)$ furnishes an example of quantum thetas, studied in [Ma1] – [Ma4]. There, especially in [Ma3], a theory of the functional equations of the type (0.2)–(0.3) is developed, applicable to the quantum tori over p -adic fields as well.

In this paper, we propose an analog of the functional equation (0.4) corresponding this time to the change of the quantization matrix $A \mapsto A^{-1}$. The noncommutative geometric context replacing the classical isomorphism of triples $(\mathcal{T}, \mathcal{L}, \theta)$ invoked above, involves now the (strong) Morita equivalence of the relevant quantum tori, compatible complex structures on these quantum tori, and theta vectors in the respective projective bimodule: see [PoS] and the references therein. The whole emerging picture is surprisingly parallel to the classical one.

0.2. Plan of the paper. In §1, we recall the basic definitions related to various Heisenberg groups we use in this paper, sketch their representation theory, and reproduce the description of quantum thetas given in [Ma3]. In §2, we recall how Heisenberg representations produce projective modules over quantum tori via lattice embeddings, and sum up the main properties of Rieffel’s scalar products in this context. In §3, we elaborate and prove the statements in (i), (ii) above for vector Heisenberg groups and their extensions by finite groups. The last subsection sketches some suggestions for further research.

§1. Heisenberg groups and their representations

1.1. Central extensions. Let \mathcal{K} (resp. \mathcal{Z}) be an abelian group written additively (resp. multiplicatively). Consider a function $\psi : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{Z}$. Then the following conditions (a) and (b) are equivalent:

(a) $\psi(0, 0) = 1$ and ψ is a cocycle, that is, for each x, y, z we have

$$\psi(x, y)\psi(x + y, z) = \psi(x, y + z)\psi(y, z). \quad (1.1)$$

(b) The following composition law on $\mathcal{G} := \mathcal{Z} \times \mathcal{K}$ turns $\mathcal{G} = \mathcal{G}(\mathcal{K}, \psi)$ into a group with identity $(1, 0)$:

$$(\lambda, x)(\mu, y) := (\lambda\mu\psi(x, y), x + y). \quad (1.2)$$

Moreover, if (a), (b) are satisfied, then the maps $\mathcal{Z} \rightarrow \mathcal{G} : \lambda \mapsto (\lambda, 0)$, $\mathcal{G} \rightarrow \mathcal{K} : (\lambda, x) \mapsto x$, describe \mathcal{G} as a central extension of \mathcal{K} by \mathcal{Z} :

$$1 \rightarrow \mathcal{Z} \rightarrow \mathcal{G}(\mathcal{K}, \psi) \rightarrow \mathcal{K} \rightarrow 1. \quad (1.3)$$

Notice that any bicharacter ψ automatically satisfies (a). For arbitrary ψ , putting $x = 0$ in (1.1), we see that $\psi(0, y) = 1$ so that

$$(\lambda, x) = (\lambda, 0)(1, x).$$

1.1.1. Bicharacter ε . Consider any central extension (1.3), choose a set theoretic section $\mathcal{K} \rightarrow \mathcal{G} : x \mapsto \tilde{x}$ and define the map $\varepsilon : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{Z}$ by

$$\varepsilon(x, y) := \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}. \quad (1.4)$$

Then ε is a bicharacter which does not depend on the choice of a section and which is antisymmetric: $\varepsilon(y, x) = \varepsilon(x, y)^{-1}$, $\varepsilon(x, x) = 1$. In particular, if $K \subset \mathcal{K}$ is a subgroup liftable to \mathcal{G} , then K is ε -isotropic.

For the group $\mathcal{G}(\mathcal{K}, \psi)$, choosing $\tilde{x} = (1, x)$, we find

$$\varepsilon(x, y) = \frac{\psi(x, y)}{\psi(y, x)}, \quad (1.5)$$

and if ψ itself is an antisymmetric bicharacter, then $\varepsilon(x, y) = \psi(x, y)^2$.

1.1.2. Example: Heisenberg groups of quantum tori and quantum theta functions. Let H be a free abelian group of finite rank written additively, k a ground field, and $\alpha : H \times H \rightarrow k^*$ a skewsymmetric pairing. The quantum torus $T(H, \alpha)$ with the character group D and quantization parameter α is represented by an algebra generated by a family of formal exponents $e(h) = e_{H, \alpha}(h)$, $h \in H$, satisfying the relations

$$e(g)e(h) = \alpha(g, h)e(g + h). \quad (1.6)$$

In particular, $T(H, 1)$ is an algebraic torus, spectrum of the group algebra $k[e(h) \mid h \in H]$ of H . The group of its points $x \in T(H, 1)(k) = \text{Hom}(H, k^*)$ acts upon functions on $T(H, \alpha)$ mapping $e_{H, \alpha}(h)$ to $x^*(e_{H, \alpha}(h)) := h(x)e_{H, \alpha}(h)$ where $h(x)$ denotes the value of the character $e(h)$ at x .

The Heisenberg group of $T(H, \alpha)$ introduced in [Ma3] and denoted there $\mathcal{G}(H, \alpha)$ consists of all maps of the form

$$\Phi \mapsto c e_{H, \alpha}(g) x^*(\Phi) e_{H, \alpha}(h)^{-1}, \quad c \in k^*; \quad x \in T(H, 1)(k); \quad g, h \in H. \quad (1.7)$$

Any such map has a unique representative of the same form in which $h = 0$ (“left representative”). Writing this representative as $[c; x, g]$ we get the composition law

$$[c'; x', g'] [c; x, g] = [c'c g(x') \alpha(g', g); x'x, g' + g]. \quad (1.8)$$

In other words, this group is the central extension of $\text{Hom}(H, k^*) \times H$ by k^* corresponding to the bicharacter

$$\psi((x', g'), (x, g)) = g(x') \alpha(g', g) \quad (1.9)$$

and having the associated bicharacter

$$\varepsilon((x', g'), (x, g)) = g(x')g'(x)^{-1}\alpha^2(g', g). \quad (1.10)$$

In particular, if a subgroup $B \subset \text{Hom}(H, k^*) \times H$ is liftable to $\mathcal{G}(H, \alpha)$, the form (1.9) restricted to B must be symmetric: this is the main part of Lemma 2.2 in [Ma3].

A lift \mathcal{L} of B to a subgroup of $\mathcal{G}(H, \alpha)$ is called a *multiplier*. The restriction to B of the form (1.9), $\langle \cdot, \cdot \rangle : B \times B \rightarrow k^*$, is called *the structure form* of this multiplier. (Formal) linear combinations of the exponents $e_{H, \alpha}$ invariant with respect to the action of $\mathcal{L}(B)$ constitute a linear space $\Gamma(\mathcal{L})$ and are called (formal) *quantum theta functions*.

1.2. Representations. Given \mathcal{K} , \mathcal{Z} , ψ and a ground field k as above, choose in addition a character $\chi : \mathcal{Z} \rightarrow k^*$. Consider a linear space of functions $f : \mathcal{K} \rightarrow k$ invariant with respect to the affine shifts and define operators $U_{(\lambda, x)}$ on this space by

$$(U_{(\lambda, x)}f)(x) := \chi(\lambda\psi(x, y))f(x + y). \quad (1.11)$$

A straightforward check shows that this is a representation of $\mathcal{G}(\mathcal{K}, \psi)$. However, it is generally reducible. Namely, suppose that there is an ε -isotropic subgroup $K_0 \subset \mathcal{K}$ liftable to $\mathcal{G}(\mathcal{K}, \psi)$. Let $\sigma : K_0 \rightarrow \mathcal{G}(\mathcal{K}, \psi)$, $\sigma(y) = (\gamma(y), y)$ be such a lift. Denote by $F(\mathcal{K}/K_0)$ the subspace of functions satisfying the following condition:

$$\forall x \in \mathcal{K}, y \in K_0, (U_{(\gamma(y), y)}f)(x) := \chi(\varepsilon(x, y))f(x), \quad (1.12)$$

or, equivalently,

$$\forall x \in \mathcal{K}, y \in K_0, f(x + y) = \chi(\gamma(y)^{-1}\psi(y, x)^{-1})f(x). \quad (1.13)$$

This subspace is invariant with respect to (1.11).

Formula (1.13) shows that if we know the value of f at a point x_0 of \mathcal{K} , it extends uniquely to the whole coset $x_0 + K_0$, hence the notation $F(\mathcal{K}/K_0)$ suggesting “twisted” functions on the coset space \mathcal{K}/K_0 .

Clearly, a minimal subspace of this kind is obtained if we choose for K_0 a maximal isotropic subgroup.

1.3. Locally compact abelian topological groups. The formalism briefly explained above is only an algebraic skeleton. In the category of *LCAb* of locally compact abelian topological groups and continuous homomorphisms, with properly adjusted definitions, one can get a much more satisfying picture.

First of all, choose $\mathcal{Z} := \mathbf{C}_1^* = \{z \in \mathbf{C}^* \mid |z| = 1\}$. This is a *dualizing object*: for each \mathcal{K} in *LCAb* there exists the internal $\text{Hom}(\mathcal{K}, \mathcal{Z})$ object, called the character

group $\widehat{\mathcal{K}}$, and the map $\mathcal{K} \mapsto \widehat{\mathcal{K}}$ extends to the equivalence of categories $LCAb \rightarrow LCAb^{op}$ (Pontryagin's duality).

Let now ψ be a continuous cocycle $\mathcal{K} \times \mathcal{K} \rightarrow \mathcal{Z}$ so that ε is a continuous bicharacter. Call the extension $\mathcal{G}(\mathcal{K}, \psi)$ a *Heisenberg group*, if the map $x \mapsto \varepsilon(x, *)$ identifies \mathcal{K} with $\widehat{\mathcal{K}}$.

Choose $k = \mathbf{C}$, and χ continuous. The formula (1.11) makes sense e.g. for continuous functions f . Especially interesting, however, is the representation on $L_2(\mathcal{K})$ which makes sense because the operators (1.11) are unitary with respect to the squared norm $\int_{\mathcal{K}} |f|^2 d\mu_{Haar}$. Of course, square integrable functions cannot be evaluated at points, so that $f(x + y)$ in (1.11) must be understood as the result of shifting f by $y \in \mathcal{K}$; similar precautions should be taken in the formula (1.13) defining now the space $L_2(\mathcal{K}/K_0)$ where K_0 is a closed isotropic subgroup (it is then automatically liftable to a closed subgroup), and in many intermediate calculations. See Mumford's treatment on pp. 5–11 of [Mu] specially tailored for readers with algebraic geometric sensibilities.

The central fact of the representation theory of a Heisenberg group $\mathcal{G}(\mathcal{K}, \psi)$, $\mathcal{K} \in LCAb$, $\chi = id$, is this:

- (i) *If K_0 is a maximal isotropic subgroup, $L_2(\mathcal{K}/K_0)$ is irreducible.*
- (ii) *Any unitary representation of $\mathcal{G}(\mathcal{K}, \psi)$ whose restriction to the center is the multiplication by the identical character is isomorphic to the completed tensor product of $L_2(\mathcal{K}/K_0)$ and a trivial representation. In particular, representations upon $L_2(\mathcal{K}/K_0)$ corresponding to different choices of K_0 are isomorphic.*

For example, if $\mathcal{K} = K_0 \times \widehat{K}_0$ is a direct product of two maximal isotropic real spaces, it has also maximal isotropic subgroups which are sublattices in \mathcal{K} , and the respective models of the Heisenberg representation are connected by a non-trivial isomorphism.

1.4. Variants and complements. The category $LCAb$ offers a clear-cut case of the representation theory of Heisenberg groups whose further axiomatization seems elusive. Nevertheless, the following general features of this case persist in one form or another when one replaces abelian groups by group schemes, objects of an abelian category etc.:

- (A) *\mathcal{Z} must be a dualizing object; Heisenberg groups are singled out among other central extensions by the condition that ε identifies \mathcal{K} with the dual object.*
- (B) *A Heisenberg group has an essentially unique representation upon twisted functions on \mathcal{K}/K_0 where K_0 is a maximal ε -isotropic subgroup.*

Even in $LCAb$ and for the case of a real space \mathcal{K} , a meaningful and important variation of the principle (B) occurs, when we allow to replace K_0 by a maximal isotropic subspace in the complexification of \mathcal{K} (see [Mu]). We will recall and use the respective construction of the Fock space model in §3 below.

It might be interesting to work out a similar formalism in a DG and derived setting. For example, in the category of abelian algebraic groups \mathbf{G}_m has many properties of a honest dualizing group, however the dual object for an abelian variety A is $\text{Ext}^1(A, \mathbf{G}_m)$ rather than $\text{Hom}(A, \mathbf{G}_m)$.

§2. Quantum tori and projective modules

2.1. Embedded lattices and tori. In this section \mathcal{K} denotes an object of $LCAb$, ψ is a bicharacter of \mathcal{K} such that ε (cf. (1.5)) identifies \mathcal{K} with $\hat{\mathcal{K}}$. Let $\mathcal{G}(\mathcal{K}, \psi)$ be the respective Heisenberg group, central extension of \mathcal{K} by $\mathbb{Z} = \mathbf{C}_1^*$ as above.

We will call *an embedded lattice* a closed subgroup $D \subset \mathcal{K}$ such that D is a finitely generated free abelian group, whereas \mathcal{K}/D is a topological torus, i.e. a finite product of S^1 . In this section we consider only those groups \mathcal{K} which admit embedded lattices.

Consider a family of constants $c_h \in \mathbf{C}_1^*$, $h \in D$. Put

$$E(h) := (c_h, h) \in \mathcal{G}(\mathcal{K}, \psi) \quad (2.1)$$

From (1.2) we get

$$E(g)E(h) = \frac{c_g c_h}{c_{g+h}} \psi(g, h) E(g+h).$$

Assume that

$$\alpha(g, h) := \frac{c_g c_h}{c_{g+h}} \psi(g, h) \quad (2.2)$$

is a skewsymmetric pairing. Then the map $e_{D, \alpha}(h) \mapsto E(h)$ is compatible with the relations (1.6), and in particular any representation U of $\mathcal{G}(\mathcal{K}, \psi)$ induces a representation of an appropriate function algebra of the quantum torus $T(H, \alpha)$. One easily sees that any α on D can be induced from an appropriate lattice embedding of D ; one can even take ψ to be a skewsymmetric bicharacter so that α will coincide with the restriction of ψ .

We will consider two function algebras of the quantum torus $T(D, \alpha)$. The algebra $C^\infty(D, \alpha)$ of smooth functions consists of infinite series $\sum_{h \in D} a_h e_{D, \alpha}(h)$ where the formal exponents satisfy (1.6), and coefficients $\{a_h \in \mathbf{C} \mid h \in D\}$ belong to the Schwarz's space $S(D)$. This algebra is endowed with involution $(\sum_{h \in D} a_h e_{D, \alpha}(h))^* = \sum_{h \in D} \bar{a}_h e_{D, \alpha}(h)^{-1}$.

The C^* -algebra $C^*(D, \alpha)$ can be defined as the universal algebra generated by the unitaries $e_{D, \alpha}(h)$ satisfying (1.6). More concretely, consider the action of $C^\infty(D, \alpha)$ upon $L_2(D)$ which is given by extending the multiplication in $C^\infty(D, \alpha)$. Complete $C^\infty(D, \alpha)$ with respect to the operator norm. The result will be $C^*(D, \alpha)$.

Alternatively, any element of $C^*(D, \alpha)$ can also be written as a formal series $\sum_{h \in D} a_h e_{D, \alpha}(h)$ but there is no transparent way to specify which sequences $\{a_h \in \mathbf{C} \mid h \in D\}$ can occur as their “noncommutative Fourier coefficients”.

A warning: the reader should not mix $\mathcal{G}(\mathcal{K}, \psi)$ with the Heisenberg group of $T(D, \alpha)$ invoked in 1.1.2: these two groups have totally different structures.

2.2. Inducing the Heisenberg representation. Let $(\lambda, x) \mapsto U_{(\lambda, x)}$ be an irreducible unitary representation of $\mathcal{G}(\mathcal{K}, \psi)$ in a Hilbert space \mathcal{H} . Since \mathcal{K} admits an embedded lattice D , it is an abelian Lie group (not necessarily connected) whose Lie algebra can be identified with the tangent space to \mathcal{K}/D at zero. The Heisenberg group $\mathcal{G}(\mathcal{K}, \psi)$ is a Lie group as well. Let L be its Lie algebra. A vector $f \in \mathcal{H}$ is called *smooth* if for any $X_1, \dots, X_n \in L$ the following expression makes sense

$$\delta U_{X_1} \circ \dots \delta U_{X_n}(f)$$

where $\delta U_X(f)$ is defined as the limit when $t \rightarrow 0$

$$\delta U_X(f) := \lim_{t \rightarrow 0} \frac{U_{\exp(tX)}f - f}{t}. \quad (2.3)$$

It is known that the space \mathcal{H}_∞ of smooth vectors is dense and the operators δU_X are skew adjoint but unbounded.

2.2.1. Theorem. *The map $e_{D, \alpha}(h) \mapsto E(h)$ induces on \mathcal{H}_∞ the structure of a finitely generated projective left $C^\infty(D, \alpha)$ -module.*

This module has an additional structure: scalar product with values in $C^\infty(D, \alpha)$. Namely, assume that the scalar product $\langle \cdot, \cdot \rangle$ in \mathcal{H} is antilinear in the second argument, and put for $\Phi, \Psi \in \mathcal{H}_\infty$

$${}_D\langle \Phi, \Psi \rangle := \sum_{h \in D} \langle \Phi, e_{D, \alpha}(h)\Psi \rangle e_{D, \alpha}(h). \quad (2.4)$$

Then this formal sum lies in $C^\infty(D, \alpha)$ and has the following properties:

- (i) *Symmetry:* ${}_D\langle \Phi, \Psi \rangle^* = {}_D\langle \Psi, \Phi \rangle$.
- (ii) *(Bi)linearity:* ${}_D\langle a\Phi, \Psi \rangle = a {}_D\langle \Phi, \Psi \rangle$ for any $a \in C^\infty(D, \alpha)$.
- (iii) *Positivity:* ${}_D\langle \Phi, \Phi \rangle$ belongs to the cone of positive elements of $C^\infty(D, \alpha)$. Moreover, if ${}_D\langle \Phi, \Phi \rangle = 0$ then $\Phi = 0$.
- (iv) *Density:* The image of $\langle \cdot, \cdot \rangle$ is dense in $C^\infty(D, \alpha)$.

2.2.2. Theorem. *The completion of \mathcal{H}_∞ with respect to the norm $\|\Phi\|^2 := {}_D\langle \Phi, \Phi \rangle$ (where the rhs means the norm in $C^\infty(D, \alpha)$) is a finitely generated projective left $C^*(D, \alpha)$ -module P . The scalar product (2.4) has a natural extension to ${}_D\langle \cdot, \cdot \rangle : P \times P \rightarrow C^*(D, \alpha)$. The properties (i) – (iv) hold for this extension as well.*

2.3. Dual embedded lattices. Let $D \subset \mathcal{K}$ be an embedded lattice as in 2.1. Denote by D^\dagger the maximal closed subgroup of \mathcal{K} orthogonal to D with respect

to ε . From the Pontryagin duality it follows that $D^!$ (resp. D) can be canonically identified with the character group of \mathcal{K}/D (resp. $\mathcal{K}/D^!$) so that $D^!$ is an embedded lattice as well.

Assume moreover that ψ is an antisymmetric pairing, so that one can choose $E(h) = (1, h) \in \mathcal{G}(\mathcal{K}, \psi)$ for $h \in D$ and for $h \in D^!$ and define on \mathcal{H}_∞ the structure of $C^\infty(D^!, \alpha^!)$ -module as well where $\alpha^!$ is the pairing induced on $D^!$ by ψ . Operators $e_{D, \alpha}(h)$, $h \in D$, commute with operators $e_{D^!, \alpha^!}(g)$, $g \in D^!$.

We can consider \mathcal{H}_∞ as a right $C^\infty(D^!, \alpha^!)^{op}$ -module. Moreover, we can and will identify the latter algebra with $C^\infty(D^!, \bar{\alpha}^!)$ by $e_{D^!, \alpha^!}(h) \mapsto e_{D^!, \bar{\alpha}^!}(h)^{-1}$ and extending this map by linearity.

2.3.1. Theorem. (i) We have $\|_D \langle \Phi, \Phi \rangle\|^{1/2} = \|_{D^!} \langle \Phi, \Phi \rangle\|^{1/2}$ The completion \mathcal{H} of \mathcal{H}_∞ with respect to this norm is a projective left module over both tori $C^*(D, \alpha)$ and $C^*(D^!, \alpha^!)$, and each of these algebras is a total commutator of the other one.

(ii) Let $C^*(D^!, \bar{\alpha}^!)$ act upon \mathcal{H} on the right as explained above. Consider the analog of the scalar product (2.4)

$$\langle \Phi, \Psi \rangle_{D^!} := \frac{1}{\text{vol } \mathcal{K}/D} \sum_{h \in D} \langle e_{D^!, \alpha^!}(h) \Psi, \Phi \rangle e_{D^!, \bar{\alpha}^!}(h) \in C^*(D^!, \bar{\alpha}^!) \quad (2.5)$$

It satisfies relations similar to (i)–(iv), and moreover, for any Φ, Ψ, Ξ the following associativity relation holds:

$${}_D \langle \Phi, \Psi \rangle \Xi = \Phi \langle \Psi, \Xi \rangle_{D^!}. \quad (2.6)$$

For proofs and further generalizations, see Rieffel's paper [Ri5], in particular, sections 2 and 3.

§3. Kähler structure, theta-vectors, and quantum thetas

3.1. Case of vector Heisenberg groups. In the first half of this section, we consider the case \mathcal{K} = a real vector space, ψ an antisymmetric bicharacter with values in \mathbf{C}_1^* .

In this case ψ can be written in the form

$$\psi(x, y) = e^{\pi i A(x, y)} \quad (3.1)$$

where $A : \mathcal{K} \times \mathcal{K} \rightarrow \mathbf{R}$ is a nondegenerate antisymmetric pairing. Choosing an appropriate basis, we can identify \mathcal{K} with the space of pairs of column vectors $x = (x_1, x_2)$, $x_i \in \mathbf{R}^N$, such that

$$A(x, y) = x_1^t y_2 - x_2^t y_1$$

where x_i^t denotes the transposed row vector. We have then $\varepsilon(x, y) = e^{2\pi i A(x, y)}$. In particular, the subspace $x_2 = 0$ is a maximal ε -isotropic closed subgroup. Similarly, \mathbf{Z}^{2N} is a maximal ε -isotropic embedded lattice.

We will recall the structure of two Heisenberg representations of $\mathcal{G}(\mathcal{K}, \psi)$ using normalizations adopted in [Mu].

Model I. It consists of square integrable complex functions f on \mathbf{R}^N , the first half of \mathcal{K} , with the scalar product

$$\langle f, g \rangle := \int f(x_1) \overline{g(x_1)} d\mu_x \quad (3.2)$$

where $d\mu_x$ is the Haar measure in which \mathbf{Z}^N has covolume 1.

The action of $\mathcal{G}(\mathcal{K}, \psi)$, with central character $\chi(\lambda) = \lambda$, is given by the formula

$$(U_{(\lambda, y)} f)(x_1) = \lambda e^{2\pi i x_1^t y_2 + \pi i y_1^t y_2} f(x_1 + y_1). \quad (3.3)$$

Model II_T. The second model is actually a family of models depending on the choice of a *compatible* Kähler structure upon \mathcal{K} . A general Kähler structure on \mathcal{K} can be given by a pair consisting of a complex structure and an Hermitean scalar product H . We will call this Kähler structure compatible (with the choice (3.1)) if $\text{Im } H = A$. Such structures are parametrized by the Siegel space consisting of symmetric matrices $T \in M(N, \mathbf{C})$ with positive defined $\text{Im } T$.

In particular, the complex structure defined by T identifies \mathbf{R}^{2N} with \mathbf{C}^N via

$$(x_1, x_2) = x \mapsto \underline{x} := T x_1 + x_2, \quad (3.4)$$

and we have

$$H(\underline{x}, \underline{x}) = \underline{x}^t (\text{Im } T)^{-1} \underline{x}^* \quad (3.5)$$

where $*$ denotes the componentwise complex conjugation.

Consider the Hilbert space \mathcal{H}_T of holomorphic functions on $\mathbf{C}^N = \mathcal{K}$ consisting of the functions with finite norm with respect to the scalar product

$$\langle f, g \rangle_T := \int_{\mathbf{C}^g} f(\underline{x}) \overline{g(\underline{x})} e^{-\pi H(\underline{x}, \underline{x})} d\nu \quad (3.6)$$

where $d\nu$ is the translation invariant measure making \mathbf{Z}^{2N} a lattice of covolume 1 in \mathbf{R}^{2N} .

For $(\lambda, y) \in \mathcal{G}(\mathcal{K}, \psi)$ and a holomorphic function f on \mathcal{K} , put

$$(U'_{(\lambda, y)} f)(\underline{x}) := \lambda^{-1} e^{-\pi H(\underline{x}, \underline{y}) - \frac{\pi}{2} H(\underline{y}, \underline{y})} f(\underline{x} + \underline{y}). \quad (3.7)$$

A straightforward check shows that these operators are unitary with respect to (3.6), and moreover, that they define a representation of $\mathcal{G}(\mathcal{K}, \psi)$ in \mathcal{H}_T corresponding to the character $\chi(\lambda) = \lambda^{-1}$ of \mathbf{C}_1^* , in the sense of formula (1.11). This is (a version of) the classical Fock representation.

It turns out that this representation is irreducible and thus is an (antidual) model of the Heisenberg representation.

The proof of irreducibility spelled out in [Mu] involves constructing *vacuum vectors* in \mathcal{H}_T which in this model turn out to be simply constant functions. Translated via canonical (antilinear) isomorphism into other models they look differently, for example they become (proportional to) a “quadratic exponent” $f_T := e^{\pi i x_1^t T x_1}$ in Model I (i. e. $L_2(\mathbf{R}^{2N}/\mathbf{R}^N)$) or to an essentially classical theta-function $e^{\pi i x_1^t \underline{x}} \vartheta(\underline{x}, T)$ in $L_2(\mathbf{R}^{2N}/\mathbf{Z}^{2N})$. They are called “theta-vectors” in [Sch]. For details, see the Theorem 2.2 in [Mu] and the discussion around it.

3.2. Scalar product $_D \langle *, * \rangle$. We will now use Model I in order to induce projective modules over toric algebras corresponding to lattice embeddings $D \subset \mathcal{K}$ as in 2.2 above. Since our ψ is already antisymmetric, we may and will put $c_h = 1$ in (2.1), so that α is the restriction of ψ to D . The main result of this subsection is the following calculation.

3.2.1. Theorem. (i) *We have*

$$_D \langle f_T, f_T \rangle = \frac{1}{\sqrt{2^N \det \operatorname{Im} T}} \sum_{h \in D} e^{-\frac{\pi}{2} H(\underline{h}, \underline{h})} e_{D, \alpha}(h). \quad (3.8)$$

Moreover,

$$\Theta_D := \sum_{h \in D} e^{-\frac{\pi}{2} H(\underline{h}, \underline{h})} e_{D, \alpha}(h)$$

is a quantum theta function in the ring $C^\infty(D, \alpha)$ satisfying the following functional equations:

$$\forall g \in D, \quad C_g e_{D, \alpha}(g) x_g^*(\Theta_D) = \Theta_D, \quad (3.9)$$

where

$$C_g = e^{-\frac{\pi}{2} H(\underline{g}, \underline{g})}, \quad x_g^*(e_{D, \alpha}(h)) = e^{-\pi H(\underline{g}, \underline{h})} e_{D, \alpha}(h). \quad (3.10)$$

(ii) *We have*

$$\Theta_D \mathbf{1} = \sum_{h \in D} e^{-\pi H(\underline{h}, \underline{h}) - \pi H(\underline{x}, \underline{h})} \quad (3.11)$$

where $\mathbf{1}$ is the vacuum vector in the model II_T represented by the function identically equal to 1.

Remarks. (i) In the language of [Ma3] and 1.1.2 above, (3.8) means that Θ_D is invariant with respect to the multiplier $D \subset \mathcal{G}(D, \alpha)$ where D is embedded in the Heisenberg group of our torus via $g \mapsto [C_g; x_g, g]$ (left representatives).

(ii) The function $\Theta_D \mathbf{1}$ is *complex conjugate* to the classical theta function corresponding to a principal polarization of the complex torus \mathbf{C}^g/D . Notice that this complex torus is embedded into (the space of points of) the algebraic torus $T(D, 1)(\mathbf{C}) = \text{Hom}(D, \mathbf{C}^*)$ as its compact subtorus $\text{Hom}(D, \mathbf{C}_1^*)$.

Proof of Theorem 3.2.1. (i) We have, using (2.4),

$$_D \langle f_T, f_T \rangle = \sum_{h \in D} \langle f_T, U_{(1,h)} f_T \rangle e_{D,\alpha}(h).$$

From (3.3) we find

$$(U_{(1,h)} f_T)(x_1) = e^{\pi i (x_1^t + h_1^t) T(x_1 + h_1) + 2\pi i x_1^t h_2 + \pi i h_1^t h_2}. \quad (3.12)$$

so that in view of (3.2)

$$\langle f_T, U_{(1,h)} f_T \rangle = \int_{\mathbf{R}^N} e^{\pi i x_1^t T x_1 - \pi i (x_1^t + h_1^t) \bar{T}(x_1 + h_1) - 2\pi i x_1^t h_2 - \pi i h_1^t h_2} d\mu. \quad (3.13)$$

It remains to calculate the Gaussian integral in (3.13).

The exponent under the integral sign can be represented as $e^{-\pi(q(x_1) + l_h(x_1) + c_h)}$ where

$$q(x_1) = 2x_1^t \text{Im } T x_1, \quad l_h(x_1) = 2ix_1^t (\bar{T}h_1 + h_2), \quad c_h = ih_1^t (\bar{T}h_1 + h_2). \quad (3.14)$$

Notice that $Th_1 + h_2 = \underline{h}$ in the notation (3.4), so that $\bar{T}h_1 + h_2 = \underline{h}^*$ where $*$ denotes the componentwise complex conjugation.

We can solve for $\lambda_h \in \mathbf{C}^N$ the equation

$$q(x_1 + \lambda_h) - q(\lambda_h) = q(x_1) + l_h(x_1)$$

We get

$$\lambda_h = \frac{i}{2} (\text{Im } T)^{-1} \underline{h}^*. \quad (3.15)$$

Therefore

$$\int e^{-\pi(q(x_1) + l_h(x_1) + c_h)} d\mu = e^{-\pi(c_h - q(\lambda_h))} \int e^{-\pi q(x_1 + \lambda_h)} d\mu = \frac{e^{-\pi(c_h - q(\lambda_h))}}{\sqrt{\det q}}. \quad (3.16)$$

Directly calculating $c_h - q(\lambda_h)$ we get

$$\frac{1}{2} \underline{h}^t (\text{Im } T)^{-1} \underline{h}^* = \frac{1}{2} H(\underline{h}, \underline{h}).$$

Moreover, $\det q = 2^N \det \operatorname{Im} T$ in view of (3.14). Hence we finally recover (3.8).

Now from (3.10) we deduce

$$C_g e_{D,\alpha}(g) x_g^* \left(\sum_{h \in D} e^{-\frac{\pi}{2} H(\underline{h}, \underline{h})} e_{D,\alpha}(h) \right) =$$

$$e^{-\frac{\pi}{2} H(\underline{g}, \underline{g})} e_{D,\alpha}(g) \sum_{h \in D} e^{-\frac{\pi}{2} H(\underline{h}, \underline{h}) - \pi H(\underline{g}, \underline{h})} e_{D,\alpha}(h) = \sum_{h \in D} e^{-\frac{\pi}{2} H(\underline{g} + \underline{h}, \underline{g} + \underline{h})} e_{D,\alpha}(g + h)$$

because $e_{D,\alpha}(g) e_{D,\alpha}(h) = e^{\pi i \operatorname{Im} H(\underline{g}, \underline{h})} e_{D,\alpha}(g + h)$. This establishes (3.9).

(ii) Formula (3.11) now follows from (3.7) and (3.8).

3.3. A functional equation for quantum thetas: the basic case. Denote now by D^\dagger the dual embedded lattice as in 2.3. From (3.1) one easily deduces that

$$D^\dagger = \{x \in \mathcal{K} \mid \forall y \in D, A(x, y) \in \mathbf{Z}\}. \quad (3.17)$$

In particular, $D^\dagger = \operatorname{Hom}(D, \mathbf{Z})$.

The following theorem is in fact a particular case of the associativity formula (2.6) written for $\Phi = \Psi = \Xi = f_T$. We replace one f_T by the vacuum vector $\mathbf{1}$ in the Model Π_T and sketch an independent proof because this explicitly shows the structure of our functional equation.

3.3.1. Theorem. *We have the following functional equation for the pair of quantum theta functions*

$$\Theta_D \mathbf{1} = \Theta_{D^\dagger} \mathbf{1}. \quad (3.18)$$

In other words,

$$\sum_{h \in D} e^{-\pi H(\underline{h}, \underline{h}) - \pi H(\underline{x}, \underline{h})} = \sum_{g \in D^\dagger} e^{-\pi H(\underline{g}, \underline{g}) - \pi H(\underline{x}, \underline{g})} \quad (3.19)$$

as functions of $x \in \mathcal{K}$.

Proof. (3.19) will follow from the Poisson summation formula if we check the following. Put

$$f_x(h) := e^{-\pi H(\underline{h}, \underline{h}) - \pi H(\underline{x}, \underline{h})}$$

considered as a function of $h \in \mathcal{K}$ (x now being a parameter). Define its Fourier transform by

$$\widehat{f}_x(g) = \int_{\mathcal{K}} f_x(h) e^{-2\pi i A(g, h)} d\nu_h.$$

Then in fact $\widehat{f}_x = f_x$.

The argument is similar to that in the proof of Theorem 3.2.1. Denote by $Q(x)$ the real positive quadratic form $H(\underline{x}, \underline{x})$ on \mathcal{K} considered as a real space. Put

$R = \operatorname{Re} T$, $S = \operatorname{Im} T$. After a somewhat tedious but straightforward calculation we find the matrix of $Q(x)$ written in real coordinates (x_1, x_2) :

$$Q(x) = x_1^t (RS^{-1}R + S) x_1 + 2 x_1^t RS^{-1} x_2 + x_2^t S^{-1} x_2 =$$

$$(x_1^t, x_2^t) \begin{pmatrix} RS^{-1}R + S & RS^{-1} \\ S^{-1}R & S^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now, find $\eta = (\eta_1, \eta_2) \in \mathbf{C} \otimes_{\mathbf{R}} \mathcal{K}$ from the equation

$$Q(h + \eta) - Q(\eta) = H(\underline{h}, \underline{h}) + H(\underline{x}, \underline{h}) + 2iA(g, h). \quad (3.20)$$

The result is

$$\eta_1 = \frac{1}{2} x_1 - \frac{i}{2} [(S^{-1}R(x_1 + g_1) + S^{-1}(x_2 + g_2))],$$

$$\eta_2 = \frac{1}{2} x_2 + \frac{i}{2} [(S + RS^{-1}R)(x_1 + g_1) + RS^{-1}(x_2 + g_2)]. \quad (3.21)$$

Then we calculate $Q(\eta)$:

$$Q(\eta) = -H(\underline{g}, \underline{g}) - H(\underline{x}, \underline{g}). \quad (3.22)$$

Finally, from (3.20) and (3.22) we find

$$\widehat{f}_x(g) = \int_{\mathcal{K}} e^{-\pi q(h+\eta) + \pi q(\eta)} d\nu_h = e^{\pi q(\eta)} = f_x(g).$$

3.4. Comment: comparison of α and α^\dagger . Choose a basis $(h_k \mid k = 1, \dots, N)$ of D and consider the skew-symmetric matrix $A := (A(h_k, h_l))$. It determines the quantization parameter α of $T(D, \alpha)$. To calculate $\overline{\alpha}^\dagger$, choose a dual basis (g_l) of D^\dagger determined by the condition $A(g_k, h_l) = \delta_{kl}$. A straightforward calculation shows that

$$A^\dagger := A(g_k, g_l) = -A^{-1}$$

so that finally $\overline{\alpha}^\dagger(g_k, g_l) = e^{\pi i (A^{-1})_{kl}}$.

The map $A \mapsto A^{-1}$ is one of the standard generators of the group $O(N, N; \mathbf{Z})/(\pm 1)$ acting on the strong Morita equivalence classes of the quantum tori: cf. [RiSch] and [Li].

3.5. Comment: invariant characterization of quantum theta functions of the form Θ_D . In the language of [Ma3], §2, Θ_D is the generator of the space $\Gamma(\mathcal{L})$ where \mathcal{L} is the ample multiplier $\mathcal{L} : D \rightarrow \mathcal{G}(D, \alpha)$, $\mathcal{L}(g) = [C_g; x_g, g]$ (left representatives), C_g and x_g being defined by (3.9).

In fact, we have to check the conditions of [Ma3], Theorem 2.4.1. The structure bilinear form of \mathcal{L} (see [Ma3], (2.2)) is

$$\langle h_1, h_2 \rangle = h_2(x_{h_1}) \alpha(h_1, h_2) = e^{-\pi H(\underline{h}_1, \underline{h}_2)} e^{\pi i H(\underline{h}_1, \underline{h}_2)} = e^{-\pi \operatorname{Re} H(\underline{h}_1, \underline{h}_2)}. \quad (3.23)$$

Since $\log |\langle h, h \rangle| = -H(\underline{h}, \underline{h})$ is a negative defined quadratic form, and the projection $D \rightarrow \mathcal{G}(D, \alpha) \rightarrow D$ is the (identical) isomorphism, \mathcal{L} is ample and $\dim \Gamma(\mathcal{L}) = 1$.

Moreover, as this calculation shows, \mathcal{L} has the following additional properties:

- (a) The structure bilinear form of \mathcal{L} is real.
- (b) There exists a Kähler structure upon $\mathbf{R} \otimes D$ consisting of a complex structure and an Hermitean form H such that

$$\langle g, h \rangle = e^{-\pi \operatorname{Re} H(\underline{g}, \underline{h})}, \quad \alpha(g, h) = e^{\pi i \operatorname{Im} H(\underline{g}, \underline{h})} \quad (3.24)$$

for all $g, h \in D$.

A converse statement is also true.

3.5.1. Theorem. *Let $T(D, \alpha)$ be a quantum torus over \mathbf{C} , with unitary quantization form α . Let $\mathcal{L} : B \rightarrow \mathcal{G}(D, \alpha)$ be an ample multiplier such that the left representative projection $B \rightarrow D$ (denoted also h^- in [Ma3]) is an isomorphism which we will use to identify B with D .*

Assume moreover that one can define a Kähler structure on $\mathbf{R} \otimes D$ such that (3.24) holds ($\langle \cdot, \cdot \rangle$ being the structure form of \mathcal{L}).

Then there exists a real space \mathcal{K} endowed with a bicharacter ψ , an compatible Kähler structure, and a lattice embedding $D \subset \mathcal{K}$ such that ψ induces α on D , and an appropriate generator of $\Gamma(\mathcal{L})$ is of the form Θ_D as above.

Proof. To see this, one should simply reverse the arguments above. Take $\mathcal{K} = \mathbf{R} \otimes D$ with the tautological embedding of D , choose the Kähler structure such that (3.24) holds and calculate the coefficients of an arbitrary generator of $\Gamma(\mathcal{L})$ as in [Ma3], (2.7). We will get the right hand side of (3.8), up to a multiplicative constant.

3.6. A generalization. In this subsection, we will generalize the Theorem 3.2.1 to the case of a lattice embedding $D \subset \mathbf{R}^{2N} \times F \times \widehat{F}$ where F is a finite group.

Define the bicharacter ψ_0 of $F \times \widehat{F}$ by

$$\psi_0((a, l), (a', l')) := l'(a). \quad (3.25)$$

Via projection, we may and will consider it as a bicharacter on $\mathbf{R}^{2N} \times F \times \widehat{F}$. Similarly, ψ from (3.1) induces a bicharacter of $\mathbf{R}^{2N} \times F \times \widehat{F}$ which we denote by

the same letter. Consider the Heisenberg group $\mathcal{G}(\mathbf{R}^{2N} \times F \times \widehat{F}, \psi\psi_0)$. The map $\mathcal{G}(\mathbf{R}^{2N}, \psi) \times \mathcal{G}(F \times \widehat{F}, \psi_0) \rightarrow \mathcal{G}(\mathbf{R}^{2N} \times F \times \widehat{F}, \psi\psi_0)$,

$$((\lambda, x), (\mu, a, l)) \mapsto (\lambda\mu, x, a, l)$$

identifies the latter group with the quotient $(\mathcal{G}(\mathbf{R}^{2N}, \psi) \times \mathcal{G}(F \times \widehat{F}, \psi_0)) / \{(\lambda, \lambda^{-1})\}$ (where the subgroup is embedded in the center in an obvious way). The Heisenberg representation \mathcal{H} of $\mathcal{G}(\mathbf{R}^{2N} \times F \times \widehat{F}, \psi\psi_0)$ is thus identified with the tensor product of the Heisenberg representations $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the two factors. We will take the Model I for \mathcal{H}_1 . For \mathcal{H}_2 , take the space of complex functions on F with the scalar product $\langle \phi, \chi \rangle := \sum_{a \in F} \phi(a) \overline{\chi(a)}$ and the action of $\mathcal{G}(F \times \widehat{F}, \psi_0)$

$$(U_{(\lambda; a, l)} \phi)(b) := \lambda l(b) \phi(a + b). \quad (3.26)$$

Let $\delta_a \in \mathcal{H}_2$ be the delta function supported by $a \in F$, and consider the vectors $f_{T,a} := f_T \otimes \delta_a$ in \mathcal{H} .

Fix a lattice embedding $D \subset \mathbf{R}^{2N} \times F \times \widehat{F}$ and denote by D_0 the kernel of the projection $D \rightarrow F \times \widehat{F}$. Since ψ_0 is not antisymmetric, we will assume that a map $D \rightarrow \mathbf{C}_1^*$, $h \mapsto c_h$ has been chosen such that c_h depends only on $h + D_0$, is 1 on D_0 , and the form on $D \times D$

$$\alpha(g, h) := \frac{c_g c_h}{c_{g+h}} \psi(g, h) \psi_0(g, h) \quad (3.27)$$

is antisymmetric.

Consider \mathcal{H} as a $C^*(D, \alpha)$ -module via $e_{D, \alpha}(h) \mapsto E(h) := c_h U_{(1, h)}$.

3.6.1. Theorem. (i) The scalar products ${}_D \langle f_{T,a}, f_{T,b} \rangle$ are quantum theta functions belonging to the space $\Gamma(\mathcal{L})$ where \mathcal{L} is the multiplier

$$D_0 \rightarrow \mathcal{G}(D, \alpha) : g \mapsto [C_g; x_g, g], \quad (3.28)$$

C_g, x_g being defined by (3.10), with H is lifted to $\mathbf{R}^N \times F \times \widehat{F}$ via projection.

(ii) The scalar products ${}_D \langle f_{T,a}, f_{T,b} \rangle$ form a basis of $\Gamma(\mathcal{L})$.

Proof. We have for $h = (h', a_h, l_h) \in D$, $h' \in \mathbf{R}^N$, $a_h \in F$, $l_h \in \widehat{F}$,

$$\begin{aligned} e_{D, \alpha}(h) f_{T,b} &= c_h U_{(1, h)} (f_T \otimes \delta_b) = \\ c_h U_{(1, h')} f_T \otimes U_{(1, a_h, l_h)} \delta_b &= c_h U_{(1, h')} f_T \otimes l_h \delta_{b-a_h}. \end{aligned} \quad (3.29)$$

Therefore,

$$\langle f_{T,a}, e_{D, \alpha}(h) f_{T,b} \rangle = \langle f_T, c_h U_{(1, h')} f_T \rangle \cdot \langle \delta_a, l_h \delta_{b-a_h} \rangle =$$

$$\frac{\bar{c}_h \bar{l}_h(a) \delta_{a+a_h, b}}{\sqrt{2^N \det \operatorname{Im} T}} e^{-\frac{\pi}{2} H(\underline{h}', \underline{h}')} \quad (3.30)$$

(cf. (3.8) and (3.26)). Hence

$${}_D \langle f_{T,a}, f_{T,b} \rangle = \frac{1}{\sqrt{2^N \det \operatorname{Im} T}} \sum_{h \in D} \bar{c}_h \bar{l}_h(a) \delta_{a+a_h, b} e^{-\frac{\pi}{2} H(\underline{h}', \underline{h}')} e_{D,\alpha}(h). \quad (3.31)$$

Denote the last sum $\Theta_{a,b}$, and take $g \in D_0$. We have

$$\begin{aligned} C_g e_{D,\alpha}(g) x_g^*(\Theta_{a,b}) &= \\ e^{-\frac{\pi}{2} H(\underline{g}', \underline{g}')} e_{D,\alpha}(g) \sum_{h \in D} \bar{c}_h \bar{l}_h(a) \delta_{a+a_h, b} e^{-\frac{\pi}{2} H(\underline{h}', \underline{h}')} e_{D,\alpha}(h) &= \\ \sum_{h \in D} \bar{c}_{g+h} \bar{l}_{g+h}(a) \delta_{a+a_{g+h}, b} e^{-\frac{\pi}{2} H(\underline{g}'+\underline{h}', \underline{g}'+\underline{h}')} e_{D,\alpha}(g+h) &= \Theta_{a,b} \end{aligned}$$

because in view of (3.27), $e_{D,\alpha}(g) e_{D,\alpha}(h) = e^{\pi i \operatorname{Im} H(\underline{g}', \underline{h}')} e_{D,\alpha}(g+h)$ whenever $g \in D_0$, $h \in D$, and moreover, $\bar{c}_h \bar{l}_h(a) \delta_{a+a_h, b}$ depends only on $h \bmod D_0$.

(ii) From [Ma3], Theorem 2.4.1, it follows that $\dim \Gamma(\mathcal{L}) = [D : D_0]$. The latter index equals $\operatorname{card} F \times \widehat{F} = (\operatorname{card} F)^2$ because $D \subset \mathbf{R}^N \times F \times \widehat{F}$ is a lattice embedding and hence the map $h \mapsto (a_h, l_h)$ must be surjective. On the other hand, when a, b run over F , the functions $\phi_{a,b} : D/D_0 \rightarrow \mathbf{C} : h + D_0 \mapsto \bar{c}_h \bar{l}_h(a) \delta_{a+a_h, b}$ span the whole space of functions. From this one derives the last statement of the Theorem.

3.7. Further problems. The picture described above is incomplete in at least two respects.

First, we did not treat theta vectors in all possible lattice embeddings, namely embeddings into self-dual locally compact groups of the form *vector space* \times *finite group* \times *torus* \times *lattice*. Besides the naturality of this question, it is necessary to understand the situation because the Heisenberg modules produced from such embeddings provide some useful generators of the strong Morita equivalence group $SO(n, n; \mathbf{Z})$, see [Li]. More generally, one can try to treat directly the scalar products of theta vectors in the toric projective modules endowed with an Hermitean connection of constant curvature. Are they all quantum thetas? Do we get in this way quantum thetas more general than those described in the Theorems 3.5.1 and 3.6.1?

Second, one should systematically study the behavior of theta vectors and their scalar products with respect to the tensor products of toric bimodules, as this was done for two-dimensional tori in [PoS].

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